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# Spontaneous magnetization of quark matter in the inhomogeneous chiral phase



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## ABSTRACT

Considering the density wave of scalar and pseudoscalar condensates, we study the response of quark matter to a weak external magnetic field. In an external magnetic field, the energy spectrum of the lowest Landau level becomes asymmetric about zero, which is closely related to chiral anomaly, and gives rise to the spontaneous magnetization. This mechanism may be one of candidates for the origin of the strong magnetic field in pulsars and/or magnetars.

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Recently, the existence of the inhomogeneous chiral phase in the QCD phase diagram has been discussed by the analysis of the effective models such as Nambu–Jona–Lasinio (NJL) model [1–3] or the Schwinger–Dyson approach [4]. In this phase, the quark condensates spatially modulate and it is very similar to the FFLO state in superconductor [5,6] or spin/charge density wave [7,8]. Here, we consider “dual chiral density wave (DCDW)” [1] among many kinds of form of the condensates: the quark condensates then take the form,

$$\Delta(\mathbf{r}) \equiv \langle \bar{\psi} \psi \rangle + i \langle \bar{\psi} i \gamma^5 \tau_3 \psi \rangle = \Delta e^{iqz}, \quad (1)$$

within the two-flavor QCD. This configuration is also obtained by embedding one of the Hartree–Fock solutions in the NJL<sub>2</sub> model, so-called chiral spiral [9,10]. Since the DCDW phase has been expected to appear in the moderate density region [1], it may be plausible that this phase is realized in neutron stars.

The effect of the magnetic field has been first discussed by Frolov et al. for the DCDW phase [11]. They have found that the spatial direction of the wavevector  $\mathbf{q}$  is favored to be parallel to the magnetic field, and the domain of the DCDW phase is much extended in the QCD phase diagram. In Ref. [12] these features arise from some topological effect through spectral asymmetry of the quark energy; quarks exhibit an interesting feature in the presence

of the magnetic field and the energy spectrum becomes asymmetric about zero. There also appear new terms in the generalized Ginzburg–Landau expansion due to spectral asymmetry, which signals the novel Lifshitz point in the QCD phase diagram. Thus, they emphasized the peculiar role of the phase degree of freedom of  $\Delta(\mathbf{r})$ .

Here we further inquire this issue. We study magnetic properties of the DCDW phase to reveal another aspect, *spontaneous magnetization* in the DCDW phase, which suggests a microscopic origin of the strong magnetic field in compact stars.

The origin of the strong magnetic field in compact stars has been one of the long-standing problems. In particular, magnetars have the huge magnetic field  $\sim 10^{15}$  G on the surface [13,14]. As a candidate of the origin, amplification of the magnetic field by the dynamo mechanism, magnetorotational instability or the hypothesis of the fossil magnetic field has been proposed so far from the macroscopic point of view. Although numerical simulations have been actively performed, no definite conclusions have been obtained. From the microscopic point of view, it has been proposed that the spontaneous magnetization emerges by spin alignment of quarks on the analogy of the electron gas [15]. However, this phase should be developed in the low density region. As another mechanism, it has been proposed that axial anomaly acting on the parallel layer of the pion domain wall produces magnetization in nuclear matter [16,17].

We use the two-flavor NJL model in the mean field approximation. It is sufficient to consider the each flavor case because La-

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grangian is diagonalized about the flavors. Then, the Dirac Hamiltonian takes the form,

$$H = -i\boldsymbol{\alpha} \cdot \mathbf{D} - 2G\gamma^0 \left[ \frac{1+\gamma_5}{2} \Delta(\mathbf{r}) + \frac{1-\gamma_5}{2} \Delta^*(\mathbf{r}) \right] \quad (2)$$

with the covariant derivative,  $\mathbf{D} = \nabla + ie_f \mathbf{A}$ . Taking the external magnetic field  $\mathbf{B}$  along the  $z$  axis, the energy spectrum constitutes the Landau levels [11],

$$E_{p_z n \zeta}^f = \epsilon \sqrt{\left( \zeta \sqrt{p_z^2 + m^2} + q/2 \right)^2 + 2|e_f B|n} \quad (n \neq 0), \quad (3)$$

$$E_{p_z \epsilon} = \epsilon \sqrt{p_z^2 + m^2} + q/2 \quad (n = 0), \quad (4)$$

with  $\zeta = \pm 1$ , which denotes the spin polarization. For  $m \leq q/2$ , the sign of  $\epsilon$  does not always correspond to the particle or anti-particle state because the lower branch of the lowest Landau level (LLL) ( $n = 0$ ) is not always negative. In the higher Landau levels (hLLs) ( $n \neq 0$ ), there are four energy branches. On the other hand, LLL has only two energy branches and becomes asymmetric about zero. As a result, the thermodynamic potential takes the form in the two-flavor case,

$$\Omega(\mu, T, B; m, q) = \frac{m^2}{4G} + N_c \sum_{f=u,d} \Omega_f, \quad (5)$$

where

$$\Omega_f = -\frac{|e_f B|T}{4\pi} \int \frac{dp_z}{2\pi} \sum_k \left\{ \sum_{n,\zeta,\epsilon} \ln \left[ \omega_k^2 + (E_{p_z n \zeta \epsilon}^f - \mu)^2 \right] + \sum_{\epsilon} \ln \left[ \omega_k^2 + (E_{p_z \epsilon} - \mu)^2 \right] \right\}, \quad (6)$$

with the Matsubara frequency,  $\omega_k = (2k+1)\pi T$ .

To investigate the response of quark matter to the weak magnetic field  $B$ , the thermodynamic potential is expanded about  $B$ ,

$$\Omega(\mu, T, B; m, q) = \Omega^{(0)}(\mu, T; m, q) + eB \Omega^{(1)}(\mu, T; m, q) + \dots, \quad (7)$$

where  $e$  denotes the elementary charge. It should be legitimate as far as  $eB/\mu^2, eB/T^2 < 1$ . For  $\mu$  or  $T$  being close to zero, it should be considered that the limit of  $B \rightarrow 0$  is taken before  $\mu$  or  $T \rightarrow 0$ .

Since the vacuum part in  $\Omega^{(0)}$  has divergence, it must be regularized by, e.g., the proper time regularization (PTR) [1]. LLL contributes only to  $\Omega^{(1)}$  because the energy spectrum does not depend on  $B$  and the  $B$  dependence only emerges through the Landau degeneracy factor. On the other hand, hLLs contribute to the all order terms of  $B$ .

The magnetization can be deduced from the thermodynamic potential as,

$$M = -\frac{\partial \Omega_{\min}(\mu, T, B)}{\partial B}, \quad (8)$$

where  $\Omega_{\min}$  represents the minimized thermodynamic potential about the order parameters and only depends on  $\mu, T$  and  $B$ . Taking the limit,  $B \rightarrow 0$ , we find the spontaneous magnetization in the form,

$$M_0 = -e\Omega^{(1)}(\mu, T; m = m^{(0)}, q = q^{(0)}), \quad (9)$$

where  $m^{(0)}$  and  $q^{(0)}$  represent the minimal values for  $\Omega^{(0)}$ . In the following, we will figure out the peculiar role of LLL and show that it leads to spontaneous magnetization.

For the evaluation of  $\Omega^{(1)}$ , we must carefully treat the effect of chiral anomaly. According to Refs. [18,19], spectral asymmetry generally gives rise to anomalous particle number,

$$N_{\text{anom}} = -\lim_{s \rightarrow +0} \frac{1}{2} \sum_k \text{sign}(\lambda_k) |\lambda_k|^{-s}, \quad (10)$$

where  $\lambda_k$  is the eigenvalue of the arbitrary Dirac Hamiltonian. Spectral asymmetry is ill-defined as it is and needs a proper regularization without violating the gauge invariance. In the DCDW phase, LLL exhibits spectral asymmetry to induce anomalous particle number proportional to  $B$  [12]. Then, the LLL contribution in  $\Omega^{(1)}$  can be decomposed into three terms,

$$\Omega^{(1),\text{LLL}} = \Omega_{\text{vac}}^{(1),\text{LLL}} + \Omega_{\mu}^{(1),\text{LLL}} + \Omega_T^{(1),\text{LLL}}, \quad (11)$$

where

$$\Omega_{\text{vac}}^{(1),\text{LLL}} = -\frac{N_c}{4\pi} \int \frac{dp_z}{2\pi} \sum_{\epsilon} |\omega_{\epsilon}|, \quad (12)$$

$$\Omega_{\mu}^{(1),\text{LLL}} = -\frac{N_c}{2\pi} \int \frac{dp_z}{2\pi} \sum_{\epsilon} (\mu - \omega_{\epsilon}) \theta(\omega_{\epsilon}) \theta(\mu - \omega_{\epsilon}) + \frac{\mu N_c}{4\pi} \eta_H, \quad (13)$$

$$\Omega_T^{(1),\text{LLL}} = -\frac{N_c T}{2\pi} \int \frac{dp_z}{2\pi} \sum_{\epsilon} \ln \left( 1 + e^{-\beta|\omega_{\epsilon} - \mu|} \right), \quad (14)$$

with  $\omega_{\epsilon} = \epsilon \sqrt{p_z^2 + m^2} + q/2$ . The density dependent term  $\Omega_{\mu}^{(1),\text{LLL}}$  includes the anomalous contribution,  $\frac{\mu N_c}{4\pi} \eta_H$ , caused by spectral asymmetry. The  $\eta$ -invariant,  $\eta_H$ , renders

$$\eta_H \equiv \lim_{s \rightarrow +0} \int \frac{dp_z}{2\pi} \sum_{\epsilon} |\omega_{\epsilon}|^{-s} \text{sign}(\omega_{\epsilon}) = \begin{cases} -\frac{q}{\pi} & (m > q/2), \\ -\frac{q}{\pi} + \frac{2}{\pi} \sqrt{q^2/4 - m^2} & (m < q/2). \end{cases} \quad (15)$$

When  $m > q/2$ , this quantity agrees with the contribution of the chiral anomaly represented by the Wess–Zumino–Witten (WZW) term [16]. The WZW term does not depend on  $m$  but it vanishes in the limit,  $m \rightarrow 0$ . The contribution of hLLs to  $\Omega^{(1)}$  should be carefully evaluated by expanding the thermodynamic potential with respect to  $B$  after the summation over  $n$ . Then, the hLLs contribution in  $\Omega^{(1)}$  can be similarly decomposed into three terms,

$$\Omega^{(1),\text{hLL}} = \Omega_{\text{vac}}^{(1),\text{hLL}} + \Omega_{\mu}^{(1),\text{hLL}} + \Omega_T^{(1),\text{hLL}}, \quad (16)$$

which does not include the anomalous contribution since hLLs have no spectral asymmetry. We find that the three terms are the even function of  $q$ , and  $\Omega_{\text{vac}}^{(1),\text{hLL}} = -\Omega_{\text{vac}}^{(1),\text{LLL}}$ . Thus  $\Omega^{(1)} = \Omega^{(1),\text{LLL}} + \Omega^{(1),\text{hLL}}$  does not diverge without any regularization and renders

$$\Omega^{(1)} = \frac{\mu N_c}{4\pi} \eta_H - \frac{N_c}{4\pi} \int \frac{dp_z}{2\pi} \sum_{\epsilon} \sum_{\tau=\pm 1} \tau (\mu - \tau \omega_{\epsilon}) \theta(\tau \omega_{\epsilon}) \theta(\mu - \tau \omega_{\epsilon}) - \frac{N_c T}{4\pi} \int \frac{dp_z}{2\pi} \sum_{\epsilon} \sum_{\tau=\pm 1} \tau \ln \left( 1 + e^{-\beta|\omega_{\epsilon} - \tau \mu|} \right). \quad (17)$$

The first term can be interpreted as the contribution of anomaly and the second and third terms as the contribution of valence quarks. Note that the even function of  $q$  in Eq. (11) is completely canceled by the corresponding one in Eq. (16) to make  $\Omega^{(1)}$  the odd function of  $q$ . It vanishes in the limit:  $m \rightarrow 0$ , which behavior

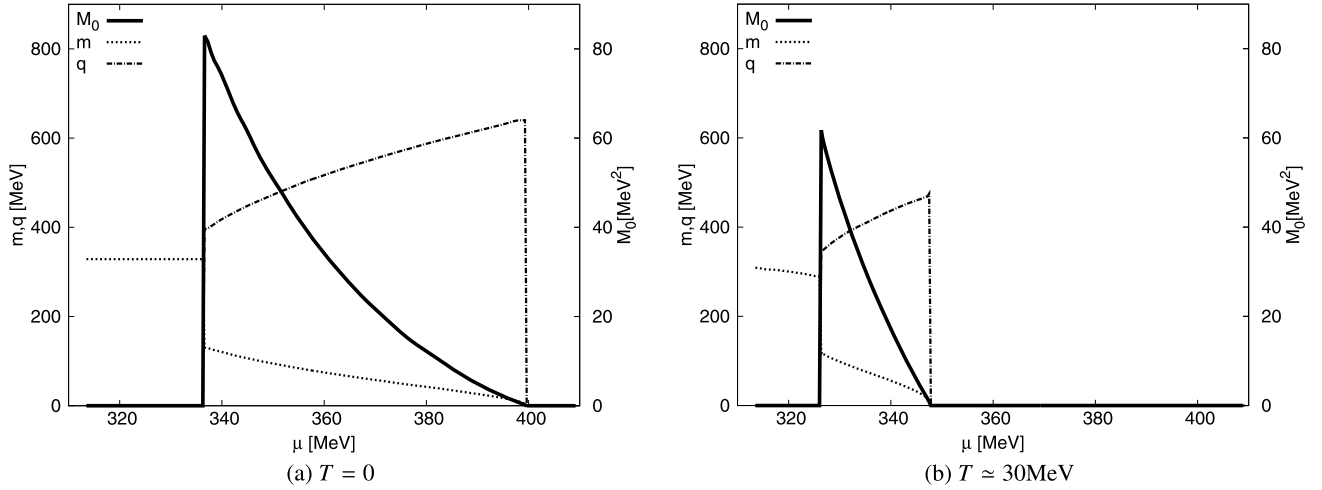


Fig. 1. The chemical potential dependence of the order parameters and spontaneous magnetization at  $B = 0$ .

may be physically reasonable because there should be no condensate and  $q$  should be redundant in this limit.<sup>1</sup> Consequently we can see that  $\Omega^{(1)}$  emerges only when  $m \neq 0$ ,  $q \neq 0$ . In other words, quark matter has the spontaneous magnetization only in the DCDW phase.

Note that the spontaneous magnetization takes a special form for some peculiar values of the parameters,  $m$ ,  $q$ , as well as  $\mu$ ,  $T$ . At  $\mu < m - q/2$  and  $T = 0$ , where is no valence quark, the magnetization only comes from the WZW term,  $M_0 = \frac{\mu N_c q}{4\pi^2}$ , argued in Ref. [16]. Unfortunately, such a situation is not realized in the present calculation because there always exist valence quarks in the DCDW phase, which is given rise to by the nesting effect of the Fermi surface [1,7,22]. On the other hand, when  $m$  becomes small compared to  $\mu$  or  $T$  but  $q$  still is not small,  $M_0$  is evaluated to be

$$M_0 = -e \frac{N_c}{8\pi^2} \sum_{\sigma=\pm 1} \sigma \operatorname{Re} \psi \left( \frac{1}{2} + i \frac{q/2 - \sigma \mu}{2\pi T} \right) m^2 + \mathcal{O}(m^4), \quad (18)$$

where  $\psi$  is the digamma function. Then, the leading order about  $m$  is  $m^2$  by the symmetry,  $m \rightarrow -m$ .

The spontaneous magnetization is numerically evaluated at  $T = 0$  and  $T \simeq 30$  MeV, for example. In this calculation, we choose  $\Lambda = 660$  MeV as the cutoff in PTR and  $G\Lambda^2 = 6.35$ , which reproduce pion decay constant  $f_\pi = 93$  MeV and constituent quark mass  $\simeq 330$  MeV in the vacuum. Fig. 1 shows the chemical potential dependence of the order parameters and the spontaneous magnetization  $M_0$ . There are three phases, the DCDW phase ( $m \neq 0$ ,  $q \neq 0$ ), the homogeneously chiral-broken phase ( $m \neq 0$ ,  $q = 0$ ) and the chiral-restored phase ( $m = 0$ ). We can see that  $M_0$  becomes nonzero only in the DCDW phase and has discontinuity at the transition point from the homogeneously chiral-broken phase. At this point, the order parameters also discontinuous, which implies the

phase transition is of the first order. On the other hand, near another transition point to the chiral-restored phase, the spontaneous magnetization is proportional to  $(m^{(0)})^2$ . Furthermore, as temperature increases, the spontaneous magnetization decreases and the region of magnetized phase gets narrower.

Considering a sphere of the uniform magnetization  $M_0$ , the magnetic field,  $B = \frac{8\pi}{3} M_0$  is produced on the surface. For the sphere of quark matter with a constant density in the DCDW phase at zero temperature, the magnetic field is estimated  $B \sim 10^{16}$  G on the surface, which may be comparable with the observation of magnetars. When there may be some residual magnetic field ( $B_{\text{res}}$ ) inherited from the progenitor stars, the correction to the magnetization  $\Delta M$  must be taken into account from the next-to-leading order term in Eq. (7). However, its contribution is proved to be tiny:  $\Delta M/M_0 \sim 10^{-6}$  for  $B_{\text{res}} = O(10^{12} \text{ G})$ . Our estimate might be too rough because density is not constant in neutron stars. Finite shell structure of the DCDW phase may be another possible realization. Considering that the DCDW phase gets narrow at high temperature, magnetic field may become much lower than this estimate in hot neutron stars. Actually the spontaneous magnetization vanishes with the disappearance of the DCDW phase at  $T \sim 100$  MeV. At  $T \simeq 30$  MeV, which may be a relevant temperature in newly born neutron stars, the magnetic field remains to be a similar order of magnitude.

In Ref. [1], the expectation value of the magnetic moment,  $\langle \bar{\psi}(\mathbf{r}) \sigma^{12} \psi(\mathbf{r}) \rangle$ , is evaluated, which behaves like spin density wave and vanishes after the spatial average. This seems to contradict with the present results. However, we can see that a careful application of the Gordon identity leads to the present results. The external magnetic field minimally couples with the quark field through the covariant derivative,  $\int d^4x \bar{\psi} Q \gamma^\mu \psi A_\mu$ , in the Lagrangian, where  $Q$  is the electric charge matrix in the flavor space. According to Ref. [23], this can be decomposed into the form,

$$\int d^4x \bar{\psi} Q \gamma^\mu \psi A_\mu = \frac{B}{2m} \int d^4x \left[ \bar{\chi} Q \sigma^{21} \chi + \bar{\chi} Q 2ix D_2 \chi + \bar{\chi} Q \sigma^{23} i\gamma^5 \tau_3 q x \chi \right], \quad (19)$$

where the Landau gauge is taken,  $\mathbf{A} = (0, Bx, 0)$ , and  $\chi$  represents the quark field after the Weinberg transformation,  $\chi = e^{i\gamma^5 \tau_3 qz/2} \psi$ . Here we have used the modified Gordon decomposition by using the Dirac equation in the presence of DCDW, instead of the usual

<sup>1</sup> The important role of LLL here may remind us of the similar phenomenon of chiral separation effect (CSE) [20,21], where the axial-vector current  $\mathbf{j}_5 = e\mathbf{B}\mu/2\pi^2$  is induced from LLL and no contribution from hLLs for  $m = 0$ . This phenomenon is related to chiral anomaly. The spontaneous magnetization is also closely related to chiral anomaly through the spectral asymmetry [12] but its manifestation is different from CSE: not only LLL but also hLLs contribute to the magnetization to leave the odd power terms of  $q$ . Note that this odd power dependence of  $q$  is peculiar to chiral anomaly [12].

treatment by the free Dirac equation. Consequently, we have the expectation value of the magnetic moment,  $\langle \bar{\chi}(\mathbf{r})\sigma^{12}\chi(\mathbf{r}) \rangle$ , instead of the above one in terms of  $\psi$ , which never vanishes after the spatial average. There appear two contributions in (19) besides the magnetic moment: the second term can be interpreted as the angular momentum and the third term proportional to  $q$  comes from the operator inherent in the DCDW phase. Thus, the magnetization discussed here should be regarded as the statistical average of these operators.

We have shown that quark matter has spontaneous magnetization in the DCDW phase where the response to the external magnetic field changes from the one in the homogeneously chiral-broken phase or the chiral-restored phase, that is,  $\Omega^{(1)}$  emerges in the thermodynamic potential. In Ref. [16], they have simply discussed the spontaneous magnetization by using the WZW term in the effective Lagrangian written in terms of only mesons. However, we have seen that  $\Omega^{(1)}$  has not only the anomalous contribution closely related to the WZW term but also the contribution of valence quarks. In this sense, the fermion degrees of freedom and their dynamics are indispensable. Thus we can say that DCDW provides a realistic ground where the spontaneous magnetization is realized.

The spontaneous magnetization discussed here has an interesting feature different from the usual spin alignment [15]: the magnetization in the DCDW phase is caused by the different operators from the naive magnetic moment. Further discussions about the peculiar magnetic properties such as magnetic susceptibility or the Nambu–Goldstone mode (spin wave) will be presented in another paper.

We have emphasized that the spectral asymmetry is important for the mechanism of the spontaneous magnetization. Since the complex order parameter  $\Delta(\mathbf{r})$  is necessary for the energy spectrum to be asymmetric [12], it is conceivable that there is no spontaneous magnetization in the “real kink crystal” [9,10] phase with the real order parameter  $\Delta(\mathbf{r})$ . On the other hand, the spontaneous magnetization should emerge in the similar way in the phase with hybrid chiral condensate [24].

It has been discussed that the NJL model, in the mean field approximation, exhibits magnetic catalysis [25–27], while the recent lattice calculation shows inverse magnetic catalysis [28,29]. To explain this phenomenon an idea of the effective four-Fermi coupling has been proposed within the NJL model [30,31], where the coupling strength depends on the external magnetic field by considering the coupling of the magnetic field with the quark and gluon loops constituting the four-Fermi coupling. Even if this is the case, however, the response to the tiny external magnetic field may be little changed and our findings should not be affected.

In order to explain the strong magnetic field in neutron stars by the present mechanism, it is necessary to study the appearance of the DCDW phase in more realistic conditions such as the charge neutrality or the chemical equilibrium. It is also an open question that the DCDW phase survives when quarks have the finite current mass [32].

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